

# The Essence of Quantum Theory for Computers

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## Abstract

Quantum computers take advantage of interfering quantum alternatives in order to handle problems that might be too time consuming with algorithms based on classical logic. Developing quantum computers requires new ways of thinking beyond those in the familiar classical world. To help in this thinking, we give a description of the foundational ideas that hold in all of our successful physical models, including quantum theory. Our emphasis will be on the proper interpretation of our theories, and not just their statements. Our tact will be to build on the concept of information, which lies central to the operation of not just computers, but the Universe. For application to quantum computing, the essence of quantum theory is given, together with special precautions and limitations.

## 1 Introduction

Having a grasp on the ideas behind a theory helps to apply it correctly, to understand its limitations, and to generate new ideas. Getting a firm hold on quantum theory is not an easy task, because our experiences and even our genetic predispositions have been developed in a world in which quantum effects are largely washed out.<sup>1</sup> Remarkably, our predilection for finding logic behind the behavior of what we observe,<sup>2</sup> including that of electrons and atoms, has led us to quantum theory, a description of nature that is hard for us to conceptualize, but is logical, accurate, and explains a wide variety of phenomena with only a few statements and input.

As background to quantum theory and quantum computing, an attempt is made here to give the primitive notions and essential observations that underlie current physical theories, so that foundational ideas are explicit, and a common language is established. In our description, information storage and transfer is made central.<sup>3</sup> A short description of quantum theory follows, and then applied

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<sup>1</sup>Although these days, macroscopic quantum effects can be seen in the actions of lasers and of quantum fluids.

<sup>2</sup>Our curiosity is enhanced by genetic selection, as there is advantage to being able to make sense of what goes on around us, so that we can anticipate what might happen next.

<sup>3</sup>Traditionally, energy transfer is used to characterize interactions in current theories. However, the concept of energy is several steps removed from more basic ideas. Moreover, infor-

to quantum computing, focusing on what the theory says, and particularly does not say, in areas where conceptual difficulties have arisen.

## 2 Physical theory and reality

A physical theory is a logical model capable of making predictions of what we observe. It is judged by its accuracy in matching measurements, and by its economy, i.e. whether the proposed theory has only a few relationships and input data needed for its ability to explain observations over a wide realm.<sup>4</sup>

We should not, however, become too enamored with the auxiliary structures within a successful theory. Just as it is possible to transform, isomorphically, a logical structure into an equivalent one involving distinctly different relationships and symbols, it is also possible to so transform a physical theory. A good example is the transformation of Maxwellian electrodynamics into an action-at-a-distance form. The transformed theory, invented by Wheeler and Feynman,<sup>5</sup> no longer contains electric or magnetic fields. Even so, it makes the same predictions as Maxwell's theory.<sup>6</sup> A lesson from this example and others is that one should not endow physical meaning to all the symbols and relationships in a theory. Electric fields do not 'exist' in nature. They exist as symbols on paper and in our minds. But Maxwell's theory does make definite statements about observations using the electric field concept. Only those points in the theory that are stated as predictions can be connected to nature. In quantum theory, wave functions are clearly not physical; in general, they are complex numbers. They can also be transformed away in alternate but equivalent theories.<sup>7</sup> Rather, one should think of the symbols and relationships in a theory as *tools for making predictions*. Predictions are the touchstones in the theory. All else is ancillary.

Here is another caution: Predictions of pure counts are testable as either true or false, but predictions of continuous values will never be proved to match nature exactly, since our measuring instruments are finite. Theories which take space as continuous implicitly do so only down to the scale permitted by our instruments. There should be no implication that even continuity exists at finer scales.

Our best physical theory so far is the so-called 'Standard Model',<sup>8</sup> which

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mation processing is not only the purpose of computers, but also lies underneath all natural processes.

<sup>4</sup>In information theory terms, the information contained in the independent data explained by a theory should be much larger than the information needed to express the theory.

<sup>5</sup>John Archibald Wheeler and Richard Phillips Feynman, "Classical Electrodynamics in Terms of Direct Interparticle Action", *Reviews of Modern Physics*, **21**, pp. 425-433 (1949).

<sup>6</sup>We generally use Maxwell's theory to solve electrodynamics problems because the Wheeler-Feynman theory is a more complicated mathematical system.

<sup>7</sup>For example, Werner Heisenberg's formulation of quantum theory, shown by P.A.M. Dirac to be completely equivalent to Erwin Schrödinger's, uses no wave functions. Neither do various so-called hydrodynamic formulations, such as that of Erwin Madelung in "Quantentheorie in Hydrodynamischer Form", *Z. Phys.* **40**, pp. 322-326 (1927).

<sup>8</sup>For a personal perspective in the development of the Standard Model, see Steven Wein-

describes, with quantum field theory, all of the interactions yet detected, except for gravity. The Standard Model has made remarkable and now verified predictions and agrees with the most precise of measurements made to one part in a trillion. Even so, the theory is not tight, having many unexplained interaction strengths and masses. We expect new theories will give a deeper and simpler explanation of particles, of their interactions, and of the yet unexplored regions in nature.

In the next section, a set of tentative propositions and observations underlying all physical theories is proposed, building toward the foundations of quantum theory and application to quantum computing. Information storage and transfer will be seen to be fundamental to natural processes.

### 3 Basic properties of physical systems

The natural world is divisible into a collection of observable subsystems. Each observable subsystem will be referred to as a **physical system**. If a physical system can be further divided, the parts may be called ‘components’ of the system. The number of divisions may reach a limit.

A physical system can store **information**, taken to be an additive quantity which grows with the number of distinct ways that the system may be configured under given physical constraints. The number of ways is called the system’s ‘**multiplicity**’,  $W$ .<sup>9</sup> To be additive across independent systems, the information  $I$  in a system must be proportional to  $\ln W$ .<sup>10</sup> With  $I = \ln_2 W$ , the information is given in ‘bits’.<sup>11</sup> If the multiplicity  $W$  of a system decreases, we say the system has become more ‘**ordered**’.<sup>12</sup>

An **interaction** between two physical systems, by definition, exchanges *information* between them. An **open physical system** can interact with other systems. **Observation** is made by allowing two physical systems to interact, one of which is prepared as a measuring instrument. A **measuring instrument** is a physical system whose information gathered from an observed system is capable

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berg’s article, “The Making of the Standard Model”, *Eur.Phys.J.* **C34**, pp. 5-13 (2004).

<sup>9</sup>One of the many remarkable implications of quantum theory is that the count  $W$  can be performed over a denumerable number of quantum states of a system.

<sup>10</sup>If there were two independent systems of multiplicity  $W_1$  and  $W_2$ , then the multiplicity of both together would be  $W_1 W_2$ . The condition  $f(W_1 W_2) = f(W_1) + f(W_2)$  makes  $f(W)$  proportional to  $\ln(W)$ .

<sup>11</sup>If a given system subject to physical constraints cannot be re-configured, then that system has only one bit of information. If the system has two possible configurations, its reading transmits one bit of information, the equivalent of a yes or a no, but no more, and so forth.

<sup>12</sup>In the late nineteenth century, Ludwig Boltzmann introduced the number  $W$  (‘Wahrscheinlichkeit’), connecting it to the disorder (Clausius’ ‘entropy’,  $S$ ) of a system with  $S = k \ln W$ , where  $k$  is Boltzmann’s constant. Leó Szilárd showed that each bit of information we gather from a system and discard necessarily requires an increase in entropy of at least  $k \ln 2$ . (“On the Decrease of Entropy in a Thermodynamic System by the Intervention of Intelligent Beings”, *Zeitschrift für Physik*, **53**, pp. 840-856 (1929).) Claude Shannon developed the formalism of information theory, including information transfer in the presence of noise. (Shannon, C.E., “A Mathematical Theory of Communication”, *Bell System Technical Journal*, **27**, pp.. 379-423 & 623-656, July & October, 1948).

of being copied with a relatively high assurance. The copy will act as a **record** of the observation. Statements about a physical system are verified only by observations.<sup>13</sup> A statement about a physical system is **predictive** if it relates a number of observations of that system. A physical system is **isolatable** if the measurable effect of all interactions with other external systems can be made arbitrarily *small*.<sup>14</sup> An isolated physical system is said to be ‘closed’ when external interactions which might influence the results of intended measurements of that system are negligible.

If a set of observations of a system is found to repeat, that system can act as a **clock**, with **time** *defined* and measured by the number of repeats, each smallest repeating cycle called a **period** of the clock. If a large set of independent periodic systems, prepared in the same way, are found to consistently have the same number  $N$  of periods, these clocks are said to be ‘good’ to a precision of at least one part in  $1/N$ .

The **distance** between two interacting physical systems is defined, up to a selected constant factor, to be the minimum time needed for an observable change in one of those two systems to cause an observable effect in the other. **Space** is defined to be the set of available distances between all systems. Two systems with a finite distance between them are said to be spatially separated. If one isolatable system can be spatially separated from all others, it is **localizable**.<sup>15</sup> If  $N$  localizable systems can be spatially separated from each other by the same distance, then space has at least  $N - 1$  spatial dimensions. A system localizable in each spatial dimension can be referred to as a **body**. The **spatial coordinates** of a body are the minimal set of numbers that uniquely determine a definable **location** within the body. These coordinates are measured by one observer relative to an ‘origin’, a location used by that observer to coordinate a set of bodies. In an  $N$  dimensional space, a complete set of such coordinates for one location is denoted  $\{x^1, x^2, \dots, x^N\}$ . An **event**,  $\{x^0, x^1, x^2, \dots, x^N\}$ , specifies when and where an observation has occurred.

A **frame of reference** characterizes how one observer records events. If the spatial separation between two bodies changes with the observer’s time, we say they have relative motion. Bodies with no average motion relative to the observer are said to be stationary. The **velocity** of a body is its spatial change per unit observer’s time along each of the independent spatial directions, and the **acceleration** is the change of velocity per unit time, each measurement made in a single frame of reference.

A **particle** is a localizable physical system with some identifiable intrinsic characteristics, i.e. quantities that are independent of how the observer mea-

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<sup>13</sup> This grounding is particularly poignant in quantum theory, wherein a quantum system is described by a set of interfering possible states for each observable, with only one such state realized by observation.

<sup>14</sup> We will use the term ‘small’ for a quantity which has the property that if made smaller, there would be no significant effect.

<sup>15</sup> Defining the localizability of zero-mass particles with spin greater than  $1/2$  (in units of Planck’s constant over  $2\pi$ ), such as the photon, is tricky. For a definition, and references back to Wolfgang Pauli, see Margaret Hawton, “Photon position operator with commuting component”, *Phys.Rev. A* **59** (2), pp.954-959 (1999).

sure them. A **fundamental particle** is a particle that suffers no measurable change its intrinsic information even after engaging in all available interactions or after long times. A ‘free particle’ is a particle whose interactions with other systems can be neglected.

Recording a **complete set of observables** in a system determines (to the degree possible) the information present in that system at the time of measurement and before any further interaction with the system. The selection of observables is made such that the measurement of any one does not change the result that would be found for the measurement of any other in the selected set. Those observables that are time independent are called **conserved**.

The ‘dynamics’ of a physical system, i.e. a description of how interacting subsystems change over time, follows logical predictive schemes which reveal *cause and effect*. These schemes are most easily tested using isolatable ‘simple systems’, i.e. those with only a few discernible component subsystems and low information content. So far, all physical systems can be described by the interactions of fundamental particles in space-time.

Systems with many interacting components, called ‘complex systems’ or ‘macrosystems’, have been successfully described when those components can be tracked, or when statistical likelihood arguments become meaningful. Systematics in the behavior of complex systems make global properties referred to as **emergent relationships**. Those of thermodynamics and statistical mechanics are examples. Rules for optimal dynamics in biosystems<sup>16</sup> form others.

Some systems, through the mutual interactions of their particles, will form bound bodies, i.e. systems that retain their localized character provided external interactions are sufficiently weak. A **confined** system is one which, when initially localized in a certain volume with zero average velocity, and then left alone, will have a non-zero lower bound on the probability of being found in the initial volume later in time. The ability to create bound systems gives preference to the evolution of differentiated systems and to condensation into locally ordered subsystems. With a sufficient variety of particles and interactions, the evolution of complexity in open subsystems is natural,<sup>17</sup> including the evolution of life.

The **Universe** is defined as the collection of everything that can be observed.

## 4 Space-time as background to quantum theory

The Universe appears to have existed in a finite number of current clock periods, and the volume of our Universe apparently is also finite. There is a limit to the greatest separation between bodies. The dimension of our space is at least

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<sup>16</sup>A ‘biosystem’ is a physical system whose activities support life. A life system is one which is capable of self-replication by interactions with external systems, using information stored within the life system.

<sup>17</sup>Lars Onsager, “Reciprocal relations in irreversible processes”, I & II, *Phys.Rev.* **37** & **38**, pp. 405-426, 2265-2279 (1931); Ilya Prigogine, *Introduction to Thermodynamics of Irreversible Processes*, New York: Interscience (1955).

three.<sup>18</sup> The distance between widely separated bodies has been growing relative to the size of the smallest bodies since time started.

Observation shows that, to a good approximation, there exist **inertial frames** in which an isolated body nearby and initially stationary relative to the observer will continue to be nearly stationary. We will use the term ‘inertial observer’ for an observer in an inertial frame. In inertial frames, an interaction experienced by one body can always be associated with the effect of other local bodies. At small scales, the relationships between local events can be expressed in a form that is independent of the observer’s position, orientation, or motion relative to the events. This is the grand ‘**Principle of Relativity**’.<sup>19</sup> The Principle of Relativity allows for the existence of a finite **universal limiting speed** for all bodies.<sup>20</sup> Examination shows that our Universe has a finite limiting speed. To the precision of current measurement, the interactions due to electromagnetism and gravity carry information between bodies at the universal speed  $c$ .

In Relativity, one observer’s measure of spatial separation between two bodies is related to a combination of space and time coordinates of another observer moving relative to the first. This makes the concept of space and time inseparable, and gives utility to the idea of a *four-vector* using the coordinates in space and time for a pair of close by events, in the form  $\{dx^\mu\} = \{dx^0, dx^1, dx^2, dx^3\}$ , where  $x^0 \equiv ct$ . Any other ordered set of four quantities forms a four-vector if they transform by coordinate transformations just like  $\{dx^\mu\}$  do.

Relativity makes the small interval between two events,  $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$  invariant,<sup>21</sup> i.e. independent of the observer’s frame of reference. The set of quantities  $\{g_{\mu\nu}\}$  form what is called the **metric tensor**. Each infinitesimal space-time region within any inertial frame can be covered by an orthogonal coordinate grid, so that the metric tensor is well approximated by  $\{g_{\mu\nu}\} \approx \text{diag}\{1, -1, -1, -1\}$ . A vector ‘dual’ to  $\{dx^\mu\}$  can be defined by  $dx_\mu \equiv g_{\mu\nu}dx^\nu$ , so that  $dx_\mu dx^\mu$  is a ‘scalar’, i.e. a number whose value is independent of the

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<sup>18</sup>Three dimensions is also the minimum dimension needed to build a computer or brain having more than four devices with mutual connections. At present, there is no evidence for higher dimensions than three. The strong experimental support of our conservation laws in three dimensions suggests that if higher dimensions of space existed, matter and energy would have had extreme difficulty passing into or out of it.

<sup>19</sup>Radiation from distant galaxies and radiation left over from the hot big bang do establish a unique frame of reference, but these are taken as part of the initial conditions in dynamics and so do vitiate the relativity principle. In our Universe, the residual effects of these initial conditions on present observations of local events are often small.

<sup>20</sup>As demonstrated by Henri Poincaré in “L’état actuel et l’avenir de la physique mathématique”, St. Louis Conference, *Bulletin des sciences mathématiques* **28**, pp.302-324 (1904). Einstein’s second postulate, the constancy of the speed of light, is not needed. Relativity alone, under reasonable assumptions about how events are measured in close by inertial frames with relative motion, initially aligned, allows only one relationship between their space-time coordinates. That relationship is the Lorentz transformation, containing a fixed universal speed called  $c$ . Explicitly, if the second inertial frame moves at a speed  $v$  away along the positive  $x$ -axis of the first, then  $x_2 = (x_1 - vt_1)/\sqrt{1 - (v/c)^2}$ ,  $y_2 = y_1$ ,  $z_2 = z_1$ ,  $t_2 = (t_1 - (v/c^2)x_1)/\sqrt{1 - (v/c)^2}$ . The Galilean transformation is approached when the universal speed in the Lorentz transformation is taken much larger than the relative speeds of the observed bodies. This makes  $t_2 \cong t_1$ , so that time becomes universal in this limit.

<sup>21</sup>By convention, repeated indices, one upper, one lower, should be summed from 0 to 3.

frame of reference of the observer. The sum  $A_\mu B^\mu$  defines the scalar product of the two vectors, and the length of  $A$  is  $\sqrt{A_\mu A^\mu}$ . An important example of a four-vector is a particle's four-momentum,  $\{p^\mu\}$ , with  $cp^0$  being the energy of the particle and  $\vec{p}$  its spatial momentum. The length of  $\{p^\mu\}/c$  is the mass of the particle.<sup>22</sup>

A general coordinate transformation between frames of reference,  $x' = f(x)$ , becomes a 'Poincaré transformation' when  $x'^\mu = a^\mu_\nu x^\nu + b^\mu$  and the coefficients  $\{a^\mu_\nu\}$  satisfy  $g_{\mu\nu} a^\mu_\kappa a^\nu_\lambda = g_{\kappa\lambda}$ .<sup>23</sup> Rotations, Lorentz transformations, and displacements are included. The set  $\{a^\mu_\nu, b^\kappa\}$  forms the so-called 'Poincaré group', with the product rule  $\{a''^\mu_\nu, b''^\kappa\} = \{a'^\mu_\lambda a^\lambda_\nu, (a'^\mu_\lambda b^\lambda + b'^\kappa)\}$ .<sup>24</sup>

A body initially stationary in an inertial frame, but acted on by one other body some distance away, will accelerate. If a duplicate of the first body is weakly bound to the first, and the experiment repeated, then the acceleration of the pair will be half the rate of the single one. We say the pair has twice the '**inertial mass**' of the single body. The inertial mass of a particle is an intrinsic property.

The observation of the effects on the motion of bodies due to the acceleration of the observer's frame with respect to an inertial frame is locally indistinguishable from the effects of gravity. This is Einstein's **Equivalence Principle**. Einstein's Equivalence Principle makes inertial mass the same as '**gravitational mass**', which is the intrinsic property of a body that determines the strength of its gravitational influence on nearby systems.<sup>25</sup> The mass of any localized system (including the equivalent mass of any associated localized field energy) can be measured by using the gravitational pull that system creates on a distant mass.

The Equivalence Principle, together with the Principle of Relativity, requires that the distance measure of space-time in the presence of a gravitating body be non-Euclidean, i.e. there will be intrinsic curvature to the space-time around a body with mass, and the metric tensor  $\{g_{\mu\nu}\}$  can no longer be transformed by a coordinate choice to the form  $\{g_{\mu\nu}\} = \text{diag}\{1, -1, -1, -1\}$  in any finite region of the space near the body. However, even in the presence of mass, inertial observers will still find an approximate flat metric in their infinitesimal neighborhood.

Einstein showed that the effects of gravity due to masses could be found from conditions on the Riemannian curvature of space-time. Curvature can be characterized by the behavior of vectors as they are moved from one point to an-

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<sup>22</sup>The energy and momentum of a system are best defined, in our successful theories, through the generators of time and space translations, with a scale determined by gravity. These ideas will be presented shortly in the context of Noether's Theorem and Einstein's General Relativity Theory.

<sup>23</sup>Note that this relation makes the metric components an 'invariant tensor', in that the components take the same values after a coordinate transformation.

<sup>24</sup>Reflections are excluded by imposing  $\det |a| = 1$ . Then the transformations are called 'proper'.

<sup>25</sup>The Equivalence Principle also means that mass  $m$  can be measured in distance units by giving  $Gm/c^2$ , where  $G$  is Newton's gravitational constant that determines the strength of gravity.

other across space. Infinitesimal changes in any vector that are observed while transporting that vector along a path define the ‘covariant derivative’:  $D_\kappa A^\mu = \partial_\kappa A^\mu - \Gamma_{\nu\kappa}^\mu A^\nu$ . The changes due to the underlying geometry come from the ‘connections’  $\Gamma_{\nu\kappa}^\mu$  in the space. In Riemannian geometry, the connections are determined by gradients of the metric tensor.<sup>26</sup> The vector  $A^\mu(x_0) - \int_{x_0}^x \Gamma_{\nu\kappa}^\mu A^\nu dx^\kappa$  is said to be the components of the ‘parallel transport’ of the original vector at  $x_0$  along a particular path to  $x$ . The change  $\delta_\square A^\mu$  in the components of any vector field,  $A^\mu(x)$ , by carrying the vector in parallel transport around an infinitesimal closed loop, must be proportional to the area of the loop and the size of the original vector field. The proportionality constants in each small patch of space-time defines the curvature tensor  $\{\mathcal{R}_{\nu\kappa\lambda}^\mu\}$  in that patch, to wit:  $\delta_\square A^\mu = \mathcal{R}_{\nu\kappa\lambda}^\mu A^\nu dx^\kappa dy^\lambda$ , where the loop is given orthogonal sides  $dx^\mu$  and  $dy^\mu$ .

Einstein’s General Theory of Relativity<sup>27</sup> is the simplest of a class of theories that incorporate the Equivalence Principle and the Principle of Relativity.<sup>28</sup> Einstein discovered that in empty space, the condition on the metric curvature tensor<sup>29</sup> given by  $R_{\kappa\mu\lambda}^\mu = 0$  numerically predicts: Newtonian gravitational fields when the effects of gravity differ little from flat space; The size of the extra perihelion precession of Mercury’s orbit; The amount of the gravitational deflection of light, and; The interval for the slowing of clocks in a gravitational field. All these and more have been confirmed to the precision of current instruments.<sup>30</sup>

In both the Special and the General Theory of Relativity, time is not universal. If two good clocks are synchronized in one frame of reference, and one is set in motion relative to the other, they may differ in the number of periods each had when they are brought back together.<sup>31</sup>

In General Relativity, bodies acted on by gravity follow a ‘geodesic’, i.e. a path that makes the invariant four-dimensional distance  $\int ds$  along the path between fixed initial and final points of the motion extreme. Free particles that travel at the ultimate speed  $c$  also follow geodesics, are necessarily massless, carry no charge, and cannot spontaneously decay.<sup>32</sup>

<sup>26</sup>In the form  $g_{\mu\lambda}\Gamma_{\nu\kappa}^\lambda = (1/2)(\partial_\kappa g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\nu\kappa})$ .

<sup>27</sup>A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie”, *Annalen der Physik* **49**, pp. 50-205 (1916).

<sup>28</sup>More general theories can be constructed using higher derivatives of the metric tensor in the field equations than the second.

<sup>29</sup>The metric curvature tensor  $\{\mathcal{R}_{\nu\mu\lambda}^\mu\}$  is that part of the local curvature tensor  $\{\mathcal{R}_{\nu\kappa\lambda}^\mu\}$  due solely to changes in the metric across space-time.

<sup>30</sup>Calculations of position on Earth using Global Positioning Satellites at height  $h$  and speed  $v$  over an Earth of mass  $M_E$  and radius  $R_E$ , have Special Relativity corrections included to order  $v^2/c^2$  for the relativistic Doppler shift and General Relativity corrections included to order  $GM_E h/(c^2 R_E^2)$  for clock slowing in a gravitational field. Without these, errors in positions would be unacceptable!

<sup>31</sup>This leads to the ‘Twin Paradox’, that one twin can end up younger than the other, yet each sees the other move away and then come back. The resolution came from Einstein using his General Theory of Relativity. The difference in the time elapsed by the clocks will be the difference between the values of  $\int \sqrt{|g_{\mu\nu} dx^\mu dx^\nu|}/c$ , integrated along the path of each clock from the common starting point to the common endpoint.

<sup>32</sup>In relativistic quantum theory, no localizable charge can be carried by a massless particle with spin greater than  $1/2$ , nor can there be a localizable flow of energy and momentum for massless particles with spin greater than  $1$ . See Steven Weinberg and Edward Witten, “Limits



Einstein's General Relativity Theory describes how the classical field  $\{g_{\mu\nu}\}$  should vary over space-time. All 'dynamical fields', to be consistent with quantum theory, must have corresponding quanta.<sup>33</sup> We expect that the quantum aspects of gravity will be important near the 'Planck scale'<sup>34</sup>  $\sqrt{\hbar G/c^3} \approx 1.6 \times 10^{-35} m$ . Although this is far smaller than the regions we can explore with current accelerators, the very rarely detected ultra-high energy cosmic rays may be scattered by this quantum granularity of space.

## 5 Quantum theory

### 5.1 The essence of quantum theory

Boiled down to its essence, quantum theory follows from a prescription due to Feynman:<sup>35</sup>

For each particle that was initially observed at  $A$  and later observed at  $B$ , construct a complex number, called the **transition amplitude**, as a sum of unimodular complex numbers according to:

$$\langle B|A \rangle = N \sum_{paths} \exp(2\pi i S/\hbar) . \quad (1)$$

The factor  $N$  will be fixed by a 'normalization' condition, introduced shortly. Each exponential term in the sum has a phase given by  $2\pi S/\hbar$ . The number  $\hbar$  is called Planck's constant. The quantity  $S$  is called the action, defined by a time-integration from  $A$  to  $B$  of a function  $L$ :

$$S = \int_A^B L dt . \quad (2)$$

The Feynman sum Eq. (1) is carried over all distinct paths between  $A$  and  $B$ .<sup>36</sup>

The function  $L$ , called the Lagrangian, depends on the particle coordinates and time changes of coordinates for the possible paths between  $A$  and  $B$ . The Lagrangian is presumed known, and often can be expressed as the particle's kinetic energy minus its potential energy. Helping to strongly limit the possible Lagrangians is the imposition of the symmetries we observe, such as the Poincaré symmetry of Relativity.

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on massless particles," *Physics Letters* **B 96 (1-2)**, pp. 59-62 (1980).

<sup>33</sup>See, for example, Matvey P. Bronstein "Quantentheorie schwacher Gravitationsfelder", *Physikalische Zeitschrift der Sowjetunion* **9**, pp. 140-157 (1936). Generally, a dynamical field varies both over space and in time. Formally, fields which have a kinetic energy term in the Lagrangian for the system are dynamic.

<sup>34</sup>Max Planck, "Über irreversible Strahlungsvorgänge.Fünfte Mitteilung", *Königlich Preussische Akademie der Wissenschaften* (Berlin), Sitzungsberichte, pp. 440-480 (1899).

<sup>35</sup>Feynman began thinking of these ideas in 1942. They are described in: R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw Hill (1965).

<sup>36</sup>For an excellent description on how Feynman paths are constructed, see Hagen Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, 5th edition, World Scientific (Singapore, 2009).

By reversing the order of the time limits in the action integral Eq.(2), the phases of the Feynman amplitudes change sign, so that time reversal of a transition amplitude is equivalent to taking its complex conjugate:  $\langle B|A \rangle = \langle A|B \rangle^*$ . Let  $B$  range over all possible states into which  $A$  may evolve. Then  $\sum_B \langle A|B \rangle \langle B|A \rangle$  gives the amplitude for the state  $A$  to explore all possible alternatives but then return to itself. We can take this amplitude to be unity and thereby fix the magnitude of the normalization constant  $N$ . We will then have

$$\sum_B \langle A|B \rangle \langle B|A \rangle = \sum_B |\langle B|A \rangle|^2 = 1. \quad (3)$$

This relation makes it possible to interpret the magnitude square of the Feynman amplitude as a probability for a given transition. Doing so creates quantum theory. That's it. All of quantum mechanics follows.

In contrast to the determinism of Newtonian theory,<sup>37</sup> quantum theory gives probabilities for the result of each measurement of a system. These probabilities are not simply the result of statistics applied to events. In quantum theory, a system can be in an interfering combination of possible realizable events before one of these events is determined by interactions with another system such as by measurement.<sup>38</sup>

If one takes field quantities as a set of equivalent particle oscillators in each infinitesimal volume of space, with the field amplitudes as the particle displacements, then quantum field theory follows.

## 5.2 The classical limit

Note that the summation of unit complex numbers with wildly different phases will tend to cancel (think of adding unit vectors in a plane with arbitrary angles between them), while a collection of such complex numbers with almost the same phase tend to add coherently. This observation applied to the Feynman path sum shows how to take the classical limit, in that those paths causing the least change in the action  $S$  relative to the size of  $\hbar$  contribute the most to the probability. Classical physics includes only those paths between two events that minimize  $S$ . This is the famous 'Principle of Least Action', from which Newton's laws and Maxwell's electrodynamics can be derived, after the appropriate choice of  $L$ .<sup>39</sup> When compared with quantum theory, Newtonian theory for particles,

<sup>37</sup>The assumption that systems have a definite state of existence between interactions would follow from having only a 'single' path dominate the Feynman sum over paths.

<sup>38</sup>The fact that certain predictions of quantum theory have intrinsic probabilistic character and that the possible realizable states of a system retain strange correlations over arbitrarily long distances between particles, greatly disturbed Einstein. But John von Neumann showed that quantum theory cannot be trivially subsumed into a bigger deterministic theory. See John von Neumann. *Mathematical foundations of quantum mechanics*, Princeton University Press, (1955), Chapter 4. For more recent work, see Roger Colbeck and Renato Renner, "No extension of quantum theory can have improved predictive power", *Nature Communications* **2**, pp.411-416 (2011). So far, all careful observations are consistent with quantum theory, even ones that Einstein called 'spooky action at a distance'.

<sup>39</sup>That non-relativistic quantum mechanics has Newtonian theory as a limit is an example of the 'correspondence limit' which we impose on any new theory in order to sustain the verified

Maxwell’s electrodynamics, and statistics applied to Newtonian systems with a large number of particles are together in a realm called ‘Classical Physics’.

A ‘**classical computer**’ is a dedicated physical system which transforms a prepared initial state into a desired output state by applying the equivalent of Boolean logic in one or more steps between input and output.

### 5.3 Superposition

From the observation that the action satisfies  $S_{BA} = S_{AC} + S_{CB}$ , it follows from Eq. (1) that

$$\langle B|A \rangle = \sum_C \langle B|C \rangle \langle C|A \rangle . \quad (4)$$

Quantum amplitudes contain a linear superposition of possible intermediate states. If the allowed Feynman paths from  $A$  to  $B$  are restricted to only those that pass through two small intermediate regions, say  $C_1$  and  $C_2$ , there will be interference of the amplitudes constructed to pass through  $C_1$  with those constructed to pass through  $C_2$ . This interference can be completely destructive, so that repeated searches for a particle at  $B$  that were launched from  $A$  come up practically empty. This effect is observed, and has *no* explanation in classical particle theory. Yes, you might say, but isn’t the particle a wave? No, we never observe particles as waves. We never find a particle ‘spread out’. Rather, the *probability* of finding a particular particle *somewhere* can be spread out over space. Individual particles are *always* found localized. Quantum theory lets us calculate these new kinds of probabilities. New, because these probabilities are found by first adding complex amplitudes, a formulation for probabilities unheard of before the second decade of the 1900s. Addition of amplitudes allows for interference effects, even for a single particle. This makes the resultant probabilities an intrinsic property of the theory, and not just due to ignorance of states in a more deterministic theory.

### 5.4 Wave functions and quantum states

The Feynman transition amplitude for a particle to leave any earlier location  $A$  with coordinates  $x_0$  at time  $t_0$  and arrive at  $B$  having the location  $x$  at time  $t$  is called the **wave function** for that particle over the spatial coordinates  $x$  at the time  $t$ :

$$\psi(x, t) = \langle B(x, t) | A(x_0, t_0) \rangle . \quad (5)$$

From Eq. (3),  $\int \psi^*(x, t) \psi(x, t) dx = 1$ . The symbol  $dx$  in the integral is to be interpreted as the volume element in space. We see that  $\psi^*(x, t) \psi(x, t) dx$  is the probability of finding the particle within the volume  $dx$ . Dirac recognized that wave functions may be considered a projection of the ‘state of the system’ described by a vector denoted  $|\psi\rangle$  onto a specific state (‘eigenstate’) of position:  $\psi(x, t) = \langle x | \psi(t) \rangle$ . Each ‘**quantum state**’  $|\psi\rangle$  can be considered a vector in

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predictions of earlier observations. After all, Newton’s theory predicts natural processes quite well for massive slowly moving bodies, like baseballs, moons, and spacecraft.

a Hilbert space.<sup>40</sup> Superposition allows us to expand the quantum state into a complete set of basis states:

$$|\psi\rangle = \sum_a |a\rangle \langle a | \psi \rangle .$$

(The sum over ‘ $a$ ’ may be given continuous regions as an approximation to discrete sums which are dense in those regions.)

From the Feynman path sums, the state of a system evolves in time according to a linear transformation

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle , \quad (6)$$

where, to keep the total probability of finding the particle anywhere unity, the operation  $U$  must be unitary:  $U^\dagger U = 1$ . (The ‘dagger’ here performs a transpose-complex-conjugate operation, rather than just complex-conjugation, to include cases in which  $\psi$  is taken to have components.) The Feynman path summation divided into small time steps means we can write  $U = \exp(-i \int H dt / \hbar)$ . (The sign in the exponent is conventional. The constant  $\hbar = h/(2\pi)$ .) The operator  $H$ , called the Hamiltonian, satisfies the ‘Hermiticity condition’  $H = H^\dagger$ . In the language of Lie groups,  $H$  is a generator of time translations. For small shifts in time,  $\psi$  satisfies a linear equation:

$$i\hbar \partial_t \psi(x, t) = H\psi(x, t) . \quad (7)$$

This is a wave equation, which formed the basis of the dynamics of quantum theory originated by Schrödinger.

## 5.5 Particles in relativistic quantum theory

Our present quantum theory incorporates Einstein’s Special Theory of Relativity.<sup>41</sup> P.A.M. Dirac, recognizing that Relativity requires that physical laws be expressible with space and time on an equal footing, wrote the Hamiltonian as a linear operator in the generators of space translation, so that the wave equation took the form<sup>42</sup>

$$\sum_{\mu=0}^3 \gamma^\mu (i\hbar \partial_\mu - (e/c) A_\mu) \psi = mc\psi . \quad (8)$$

When the fields  $A_\mu$  vanish, there are plane wave solutions  $\psi \propto \exp(-ip_\mu x^\mu / \hbar)$ , so that  $g_{\mu\nu} i\hbar \partial^\mu i\hbar \partial^\nu \psi = p_\mu p^\mu \psi = m^2 c^2 \psi$ , and the  $\gamma$ ’s must satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I.$$

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<sup>40</sup>Essentially a vector space with lengths and angles defined, but possibly infinite dimensional.

<sup>41</sup>A. Einstein, “Zur Elektrodynamik bewegter Körper”, *Annalen der Physik*, **17**, pp. 891-921 (1905).

<sup>42</sup>P.A.M. Dirac, “A Quantum Theory of Electrons”, Part I & II, *Proc.Roy.Soc. (London)*, **A 17** & **A 118**, pp. 610-624 & pp. 351-361 (1928).

If we add the assumption of reflection symmetry, the  $\gamma$ 's are square matrices with even dimension at least four. Taking the  $\gamma$ 's to be dimension four, and the fields  $A_\mu$  as the electromagnetic vector potentials due to other charges, the Dirac equation very accurately describes electrons in the field of other charges, and therefore atomic structure and, in principle, all of chemistry and molecular biology. The components of the electron wave function can be decomposed into two pairs, each pair corresponding to the two possible intrinsic spin directions measurable, and the combined pair corresponding to the electron carrying positive or negative energy. As an indication of the profound reach gained by merging quantum theory and Relativity, Dirac was able to show that the electron spin and its magnetic moment followed from relativistic quantum theory, and that antimatter must exist, a prediction before anyone dreamed of the concept.

The possibility that fundamental particles can be created and destroyed is included into quantum theory by taking the particle wave functions and interacting fields as quantum fields, entering into the action  $S$  with their own dynamics. We find that if disturbed, particle pairs can even 'bubble' out of empty space. The time-and-space-reversed wave function for a particle describes the forward progression of a corresponding antiparticle. This becomes the 'CPT Theorem' in quantum field theory, referring to the operations of charge conjugation, parity transformation, and time reversal.

Quantum field theory distinguishes particles with half-odd integer spin, called 'fermions', from those with integer spin, called 'bosons'.<sup>43</sup> The quantum field in a three-dimensional space and associated with a pair of identical particles may undergo a phase change when those two particles are exchanged:  $|\psi(1, 2)\rangle = (-1)^{2s} |\psi(2, 1)\rangle$ . If the particles are fermions ( $s = 1/2, 3/2, \dots$ ) the phase change is  $-1$ , while no phase change occurs if the particles are bosons ( $s = 0, 1, 2, \dots$ ). This means no two fermions of the same type (such as electrons in atoms) can occupy the same quantum state. This is the 'Pauli Exclusion Principle'. Any number of bosons of the same type can be in the same quantum state (e.g. photons in lasers).

The fundamental particles making up the structure of materials currently appear to be three generations of the doublet electron-neutrino,<sup>44</sup> and three generations of a doublet of quarks, all fermions. The family of electrons and neutrinos are called 'leptons'. Each generation of quarks comes in one of three distinct varieties according to their 'color charge'. The bound state of a 'red', 'green', and 'blue' quark and any other 'color-neutral' combination of an odd number of quarks generates a 'baryon', such as the familiar proton and neu-

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<sup>43</sup>Particles must have quantized spin with length  $\sqrt{s(s+1)}\hbar$  and projection along some measurement axis of  $\mu\hbar$ , where  $s$  is either a half or whole integer, and  $-s \leq \mu \leq s$ . It is conventional to use the label  $s$  to characterize the particle spin, as in "The electron has spin  $1/2$ ". Particles that move at the speed of  $c$  have only two projections of their spin, called their 'helicities', either along their momentum, or in the opposite direction. The characteristic properties of particles following from relativistic quantum theory were first described by Eugene Wigner in "On Unitary Representations of the Inhomogeneous Lorentz Group", *Ann. Math.* **40** (1), pp. 149-204 (1939).

<sup>44</sup>Our observations of the sky together with General Relativistic cosmology seem not to allow more than three generations.

tron. A zoo of more fleeting particles exist, including ‘mesons’ coming from bound color-neutral quark-antiquark systems. The large family of baryons and mesons, all strongly interacting particles, are called ‘hadrons’. In the Standard Model, leptons have no direct strong interactions.

## 5.6 Interactions in quantum theory

All the observed interactions of one particle with another can be categorized by the so-called strong, electromagnetic, weak, and gravitational forces.<sup>45</sup>

The numerical strength of a particle’s interactions with other particles is always associated with an intrinsic property called its ‘charge’. For each category of interaction, there is one or more corresponding charges. If the total charge of a closed physical system is preserved during a sequence of interactions within that system, we say the charge has been ‘conserved’. In nature, all charges are quantized, i.e., they come from a countable set.<sup>46</sup>

The existence of conserved and localizable charges means one can always define an **interaction field** that has those charges as its source, using the following argument: If  $\{j^\mu\} = \{\rho c, \rho \vec{v}\}$  represents the charge density and current density for a set of charges, then the local conservation of the total charge,  $Q \equiv \int \rho d^3x$ , can be read from  $\partial_\mu j^\mu = 0$ . But this implies the existence of an ‘interaction field’  $\{F^{\mu\nu}\}$ , antisymmetric in its indices, satisfying  $\partial_\kappa F^{\kappa\nu} \propto j^\nu$ . An associated field,  $F^*_{\mu\nu} \equiv (1/2)\epsilon_{\mu\nu\kappa\lambda} F^{\kappa\lambda}$  defines a ‘dual’ conserved charge with current  $j^{*\nu} \propto \partial_\mu F^{*\mu\nu}$ .<sup>47</sup> If no such dual charge exists in a region of space, then the field  $\{F^{\mu\nu}\}$ , assumed to carry no intrinsic charge itself,<sup>48</sup> can be expressed in terms of a vector field  $\{A^\mu\}$  by  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ . The field  $\{A^\mu\}$  is called the ‘**gauge field**’ going with the corresponding charge. Gauge fields are not uniquely determined, but may be transformed into new fields  $\{A'^\mu\}$  which have the same interaction field  $\{F^{\mu\nu}\}$  by adding a gradient:  $A'^\mu = A^\mu + \partial^\mu \Lambda$ . The choice of the ‘gauge function’  $\Lambda(x)$  is open, provided  $\oint \partial_\mu \Lambda dx^\mu$  vanishes for all closed loops in regions where the gauge field acts. Theories whose predictions are independent of the choice of gauge have ‘**gauge symmetry**’.<sup>49</sup>

Conventional theory describes particle interactions by introducing interaction fields which ‘mediate’ the effect of one charge on another. We say each particle with a charge of some kind ‘creates’ an interaction field in the space around

<sup>45</sup>The electromagnetic and weak interactions were linked, principally by the work of Salam, Glashow, Weinberg, Higgs, ’t Hooft, and Veltman, from 1964 to 1975.

<sup>46</sup>Dirac showed that if magnetic monopoles exist, then electric charge must be quantized. See Paul Dirac, “Quantised Singularities in the Electromagnetic Field”, *Proc. Roy. Soc. (London)* **A 133**, pp. 60-71 (1931).

<sup>47</sup>The  $\{\epsilon_{\mu\nu\kappa\lambda}\}$  is the completely antisymmetric tensor in four dimensions, with  $\epsilon_{0123} = 1$ , called the Levi-Civita symbol. Like  $g_{\mu\nu}$ , its components are invariant under a proper Poincaré transformation.

<sup>48</sup>For the strong-interaction charges, the field  $\{F^{\mu\nu}\}$  does carry charge. It then can interact with itself, making the wave equation for dynamical field theory constructed for  $\{F^{\mu\nu}\}$  necessarily non-linear.

<sup>49</sup>The use of ‘gauge symmetry’ was introduced by Herman Weyl in his consideration of theories with invariance in the scale of length. (H. Weyl, “Gravitation und Elektrizität”, *Sitzungsber. Preuss. Akad. Wiss.*, pp. 465-480 (1918).)

it, and that field acts on other particles having the same kind of charge. In the case of electromagnetic interactions, the interaction field is  $\{F^{\mu\nu}\}$  with components that are the electric and magnetic field, while the gauge field  $\{A^\mu\}$  is called the electromagnetic vector potential. Maxwell's equations,  $\partial_\kappa F^{\nu\kappa} = (4\pi/c)j^\nu$  and  $\partial_\kappa F^{*\nu\kappa} = 0$ , then express two conditions: Electric charge is conserved locally, and there is no observable local magnetic charge. How particles react to other charges requires knowledge of the dynamics for those particles. Dynamics is incorporated into quantum theory.

In quantum theory, a second kind of gauge transformation occurs when the phase of particle wave functions are shifted. A constant shift has no observable effect. But making a shift in phase which depends on location will introduce a relative phase between wave components. If those component waves converge, their interference is observable in the associated particle probability. Now, if, along with the phase shift, a shift in the derivatives of the wave function occurs, one can make the combined shifts cancel. This is the property built into '**gauge symmetric quantum theories**'. In fact, all the interactions among fundamental particles have been found to follow from theories which satisfy gauge symmetry!

Another property of our current dynamical theories can be called the **principle of quasi-local interactions**: The known interactions of one particle with another can be described by 'quasi-local' effects, castable into a form that requires only knowledge of the fields of other particles in a small local space-time neighborhood of the affected particle. These fields are the gauge fields described above. Consider the free-electron Dirac equation  $\hbar\gamma^\mu i\partial_\mu\psi = mc\psi$ . A gauge transformation of the second kind on the wave function can be expressed as  $\psi'(x) = \exp[-i(e/(\hbar c))\Lambda(x)]\psi(x)$ . The free-particle wave equation becomes  $\gamma^\mu(i\hbar\partial_\mu - (e/c)\partial_\mu\Lambda)\psi' = mc\psi'$ . Gauge symmetry can be enforced by adding to the derivative term a gauge field  $A_\mu$  which undergoes a gauge transformation of the first kind:  $A'^\mu = A^\mu + \partial^\mu\Lambda$ . We arrive at the full Dirac equation (8). This technique for introducing interactions is referred to as the 'minimal coupling principle'.<sup>50</sup>

A marvelous theorem was derived by Emmy Noether,<sup>51</sup> who showed that symmetries of our theories based on continuous groups of transformations, such as the Poincaré group and the gauge transformations, lead to conservation laws. In the case of the Poincaré symmetry, the conserved quantities are total energy-momentum, total angular momentum, and the velocity of the center-of-energy.

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<sup>50</sup> Gauge symmetry in quantum theory can be re-expressed in terms of the action of 'co-variant' derivatives  $D_\mu = \partial_\mu + i(e/(\hbar c))A_\mu$ , acting on a quantum state for a particle. In this interpretation, the interactions arise from the behavior of quantum states by parallel transport across space. When the gauge fields themselves are taken to be operators on the internal components of a quantum state, the gauge group elements may not commute. These kind of 'non-abelian' gauge fields were introduced by C.N. Yang and R. Mills ("Conservation of Isotopic Spin and Isotopic Gauge Invariance", *Phys.Rev.* **96** (1), pp.191-195 (1954)) and are used in the Standard Model to describe interactions between fundamental particles grouped into families. For example, the quark color charge follows from an  $SU_3$  gauge symmetry.

<sup>51</sup>E. Noether, "Invariante Variationsprobleme", *Nachr. D. König. Gesellsch. D. Wiss. Zu Göttingen, Math-phys. Klasse*, pp.235-257 (1918).

An important example is the symmetry under time translation: If experiments done now with a given system have the same set of results as those done at any time later, then the system's energy is fixed. Symmetry under constant phase shifts of a lepton or baryon wave function makes lepton and baryon number conservation. Gauge symmetry makes the corresponding charges conserved.

As the gauge fields have their own dynamics, quantum theory requires that gauge fields be quantized. That means that the interactions between material particles occur only by the exchange of quanta. These quanta are necessarily bosons. For electromagnetic interactions, the gauge-field quantum is the photon. The photon at present appears to travel at the maximum speed in Relativity, has unit spin, and carries no electric charge.<sup>52</sup> For the strong interactions between quarks, the quanta of the field are called gluons. Gluons also have unit spin but they carry various 'color' charges. By having charge, gluons can directly interact with themselves, making their dynamics more complicated than for photons. For example, the gluon fields, through their self-interaction, can form flux tubes between quarks.

## 5.7 Prepared states and measurement

Each possible quantum state of a system is referred to as a 'pure quantum state', as contrasted to a 'mixed quantum state', for which we may only know probabilities for the system to be in given quantum states. A pure quantum state made from a superposition of component states is called a 'coherent quantum state' when all the phases between its various component states are known to be fixed.

A quantum system is prepared by first selecting a physical system, isolating the system from unwanted interactions, determining its initial configuration, and then stimulating or allowing the system to approach a desired initial state.

Isolating a system and determining its initial configuration are often daunting tasks. The state of most macrosystems will be practically impossible to completely specify. Some interactions, such as those from stray fields or background radiation, may be difficult or impossible to eliminate. In addition, various possible components within an isolated system can transform and evolve even when isolated. However, fundamental particles will, by definition, be stable, at least over times much longer than observational periods. Also, some bound systems of fundamental particles will be quasi-stable if the energy need to excite the system is large compared to the energies available. After isolation, a system will evolve by quantum dynamics following a unitary transformation, and may eventually become a 'steady state', i.e. one with no change in probability densities for its particles, if these were observed.

Consider the expansion of a pure quantum state into component states which

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<sup>52</sup>A particle with charge will carry energy associated with the field of that charge, and therefore, if it can be separated from other particles, must have mass, and must move slower than the universal speed  $c$ .



together span the system's Hilbert space:

$$|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle. \quad (9)$$

If the phases between two or more components of the quantum state are related, then these components are said to be in ‘**coherence**’. Quantum interference between various possible outcomes of a measurement requires some coherence in a quantum state. The states  $|\phi_1\rangle$  and  $|\phi_2\rangle$  might be two possible interfering states of a single electron, or even a trapped atom. The two states of the atom might have opposite motions, so that the wave function for each state can oscillate back and forth across the trap. Then the probability distribution of finding the atom at a specific location within the trap shows an interference pattern.<sup>53</sup> This quantum effect, however, is no different in principle than that seen as an interference pattern made by the bright spots of light on the surface of a phosphor plate, those spots produced by electrons passing through two slits in a screen, one electron at a time, and then hitting the phosphor.

Starting with a set of identically prepared systems in a coherent state represented by Eq. (9), measurements of the observable  $A$  will have an average  $\sum_{ij} \alpha_i^* \alpha_j \langle \phi_i | A | \phi_j \rangle$ . Interference will arise from terms for which  $\langle \phi_i | A | \phi_j \rangle$  are not zero for  $i \neq j$ . However, if a quantum system interacts with another system or with a measuring device, some or all of the components of residual quantum states for that system may be left with no well-defined phase relationships. This is a process of ‘**decoherence**’. During the measurement, information is transferred between the system and the measuring device, and some may be lost to the environment.

One of the important measurements of a system locates the position of particles. After a number of such measurements in each small region  $dx$  of space, we find a distribution of positions. For one particle, the wave function determines the probability density for position across space, so the distribution of measured positions is predicted to be an approximation of  $\psi(x)^* \psi(x) dx$ . The average position over all space is predicted to be  $\langle \psi | x | \psi \rangle$ . More generally, each distinct measurement of a property of the system can be associated with a Hermitian operator  $A$  that acts on wave functions for the system  $\psi$  as follows: The average value of  $A$  will be

$$\langle A \rangle = \int \psi^\dagger(x, t) A(x, \hat{p}_x) \psi(x, t) dx, \quad (10)$$

wherein  $\hat{p}_x$  is taken proportional to the space translation operator in accord with Noether's Theorem.<sup>54</sup> The operators  $A$  may also act on the spin and other components of the wave function.

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<sup>53</sup>This game was played using a Beryllium atom by Dr. Christopher Monroe and colleagues at the National Institute of Standards and Technology, Boulder, Colorado. See C. Monroe *et al.*, “A ‘Schrödinger Cat’ Superposition of an Atom,” *Science* **272**, pp. 1131-1136 (May 24, 1996). Some members of the press mis-represented the observation as indicating that one atom can be found in two places at once. For example, see Malcolm W. Browne's article “Physicists Put Atom In 2 Places At Once”, published in the New York Times, May 28, 1996.

<sup>54</sup>The proportionality constant is fixed by noting that if a free particle is left unob-

Those states  $|a\rangle$  satisfying

$$A|a\rangle = a|a\rangle$$

are called the ‘eigenstates’ of  $A$  and  $a$  an ‘eigenvalue’. For Hermitian operators  $A$ , i.e.  $A^\dagger = A$ , the eigenvalues  $a$  will be real numbers, and therefore each is a value which may result from a measurement. The elements of the set  $\{|a\rangle\}$  for distinct values  $a$  will be ‘orthogonal’, i.e.  $\langle a'|a\rangle = \delta_{a'a}$ , and ‘complete’, i.e. they span the space of possible states, expressible as  $\sum_a |a\rangle\langle a| = I$  by reading from  $|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$ . The measured values of  $A$  will have an uncertainty defined by  $\Delta A \equiv \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$ . This means that after measurement of  $A$  for a large number of identically prepared systems, the observed values will be distributed around the average with a ‘width’ of  $\Delta A$ .

After a single measurement of the observable  $A$  for a system in a pure quantum state  $|\psi\rangle$  that has  $A$  as one of its observables, one of the eigenvalues of  $A$ , say  $a$ , will be found, and the system will be left in the state  $|a\rangle$ . The effect of measurement can be represented by a ‘**projection operation**’:  $P_a \equiv |a\rangle\langle a|$ . The measurement of  $A$  has ‘collapsed’ the quantum state to  $|a\rangle \propto P_a|\psi\rangle$ . The collapse evidently does not preserve unitarity for the system, expressed by  $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$ , unless the system was already in an eigenstate of  $A$ . Quantum unitarity applies to isolated systems. The measurement process has involved another interacting system which reduced the system’s available states in a subsequent measurement.

The interaction (called the ‘coupling’) between two systems during a measurement may cause one or more of the phase differences between the components in the resulting quantum states to become indeterminate, especially likely if the measuring device is macroscopic. After the measurement, the system may be left in a ‘mixed’ state, for which only the probabilities  $p_k$  for any particular pure quantum state  $|\psi_k\rangle$  are known. Then a subsequent measurement of the observable  $A$  will have an average value of  $\langle A \rangle = \sum_k p_k \langle \psi_k | A | \psi_k \rangle$ . This expression can be usefully re-written in terms of a ‘density operator’, defined by  $\hat{\rho} \equiv \sum_k p_k |\psi_k\rangle\langle \psi_k|$ , so that  $\langle A \rangle = \text{Tr}(\hat{\rho}A)$ . In this way, the choice of the mixed state is left implicit. A pure state can then be simply characterized by  $\hat{\rho}^2 = \hat{\rho}$ .

## 5.8 Entangled states

An ‘**entangled quantum system**’, by definition, has two or more particles in a quantum state which cannot be factorized into states for each particle.<sup>55</sup> For example, if we let the quantum state  $|\mu_\kappa\rangle$  represent the electron labeled by  $\kappa$  and having a spin projection along the  $z$ -axis of  $(\mu - 1/2)\hbar$ , then one of the

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served, then within some bounded region its wave function becomes a ‘plane’ wave  $\phi \propto \exp((ip_x x - iEt)/\hbar)$ , for which  $\hat{p}_x \phi = -i\hbar \partial_x \phi$ . The order of non-commuting operators in  $A$  must be determined by physical arguments.

<sup>55</sup>There is a special caution for quantum states describing photons, in that the number of photons is not fixed, but rather has an uncertainty which increases as the phase of the electromagnetic wave becomes more definite.

possible entangled two-electron states can be written

$$|\psi_0\rangle = (1/\sqrt{2})(|0_1\rangle|1_2\rangle - |1_1\rangle|0_2\rangle),$$

which happens to have a total spin of zero.

The outcome of a measurement of one of the electrons in an entangled pair will be correlated with the outcome of measurement of the other, even when they are far apart. This kind of correlation also occurs under classical conditions. Suppose you put a jack in one envelope and a queen in another. Now send one of the envelopes to one friend, the second envelope to a second friend. If one friend opens your envelope and finds a jack, then your other friend must find a queen even before hearing from your first friend. However, there is a twist in the quantum world. Take the case when a pair of electrons is prepared in a zero total spin state along a  $z$ -axis expressed above, and then the electrons are allowed to move far apart. Next, while the electrons are in flight, have one of the distant observers rotate her electron-spin measuring apparatus away from the  $z$ -axis direction to an angle of her choosing, i.e. the first distant observer makes a ‘**delayed-choice experiment**’.<sup>56</sup> If this first observer finds an electron aligned along her new axis, then the second observer, far away, will find the other electron aligned along the negative direction of the NEW axis constructed by the first observer. Now we, on first hearing and with our classical thinking, SHOULD be surprised! Even so, this is the way nature acts. The result does NOT mean that the pair interacted after traveling apart, nor was there ‘superluminal’ transmission of information.<sup>57</sup> This suggestion of faster-than-light signaling is a misinterpretation of quantum theory, and such information transfer has not been seen. Rather, those who say so are likely to have been tripped up by picturing each unobserved electron as being localized between observations!

## 5.9 Non-classical interactions

There are interactions predicted by quantum theory without classical explanation. Yakir Aharonov and David Bohm<sup>58</sup> showed that a single electron wave which never enters a region of electric or magnetic field could never-the-less have a measurable shift in the probability of finding that electron after an electric or magnetic field changes in the excluded region. The effect occurs, for example, when the electron passes on either side but does not enter a tube where a

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<sup>56</sup> If the decision on how a component of a system is measured comes after that system has had sufficient time to cause interference between quantum alternatives for that component, then this becomes a delayed-choice experiment as introduced by John Wheeler in *Mathematical Foundations of Quantum Theory*, edited by A.R. Marlow, Academic Press (1978).

<sup>57</sup> Information transmitted by a wave disturbance that started at a certain time cannot be transferred faster than the outgoing wavefront from that disturbance. In Special Relativity, the speed of the wavefront, also called the signal speed, is always less than or equal to the universal limiting speed,  $c$ . There is no such restriction on the group velocity or the phase velocity of the wave.

<sup>58</sup> Y. Aharonov and D. Bohm, “Significance of Electromagnetic Potentials in the Quantum Theory”, *Phys.Rev.* **115**, pp. 485-491 (1959).

magnetic field is confined. The difference in phase of that wave when followed around a closed loop is given by  $e \oint A^\mu dx_\mu / (\hbar c)$ , where  $\{A^\mu\}$  is the electromagnetic potential. This is the flux of magnetic field somewhere inside the loop.<sup>59</sup> The shift in the observed interference pattern produced by the electrons when the magnetic flux is turned on has no explanation in classical physics.

Measurement of a system may disturb the system. If the measurement process transfers complete information about a system, that system will no longer contain entangled states. This effect leads to the ‘no-cloning theorem’,<sup>60</sup> the statement that a general quantum system containing some coherence cannot be identically copied. If a copy of a quantum state could be made, then we could defeat the interfering effect of measurement by first making a copy, and then measuring the copy, leaving the original system undisturbed.

The wave function for a particle confined to a fixed region of space and initially localized to a much smaller part of that region and then left with no external interaction will diffuse outward in space as time progresses. The wave for an unobserved particle will spread over the entire allowed region, and eventually the probability for finding the particle in any small location will have no measurable change in time, and its quantum wave function will be steady.<sup>61</sup>

A localized and isolated physical system will have denumerable (‘quantized’) possible values for its measurable energies and momenta. Periodicities of the wave function also enforce quantization if there is a closed path over which the corresponding particle can move. For example, periodicity in the azimuthal angle in the wave function makes the measured values of the projection of the orbital angular momentum along a measurement axis denumerable.

Suppose two observables  $A$  and  $B$  for a given system in the state  $|\psi\rangle$  are measured in a certain time order. If these two measurements are repeated for identically prepared systems, a change in the order of measurement may change the probability for finding a given value for the second observable. In general, one can show that the uncertainties satisfy  $\Delta A \Delta B \geq (1/2) |\langle AB - BA \rangle|$ . This is called the uncertainty principle of Heisenberg. If the ‘commutator’  $[A, B] \equiv (AB - BA)$  vanishes, then the observables  $A$  and  $B$  may be measured ‘simultaneously’, i.e. without the measurement of one affecting the results of measuring the second. The state of a physical system can be labeled by a set of measured values for a maximal set of mutually commuting observables that are also conserved over time.

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<sup>59</sup>Mandelstam re-expressed the local interaction with  $\{A^\mu\}$  as a non-local effect of the electric and magnetic fields, i.e. a topological effect of fields over space-time. See Stanley Mandelstam, “Quantum electrodynamics without potentials”, *Ann. Phys.* **19**, pp. 1-24 (1962).

<sup>60</sup> Wojciech Zurek, “A Single Quantum Cannot be Cloned”, *Nature* **299**, pp. 802-803 (1982); Dennis Dieks, “Communication by EPR devices”, *Physics Letters A* **92** (6), pp. 271-272 (1982).

<sup>61</sup>Steady wave functions necessarily have a sinusoidal time dependence through a factor of the form  $\exp(-i\omega t)$ , making the probability  $\psi^\dagger \psi dx$  time independent.

## 5.10 Quantum theory for complex systems

After a relaxation time for a system containing a large number of interacting particles, the most likely distribution of particles in the available quantized energy states will be those that tend to maximize, under the physical constraints, the multiplicity  $W$ , simply because as the system evolves through various configurations, it will spend most of its time in those configurations which have many ways of being constructed. One can then show the following:<sup>62</sup> Divide the system into a large number of subsystems, labeled by  $i$ , which have possible energy states equally likely to accept energy from its neighbors. (One such choice of subsystem could be the identifiable particles in the system.) Suppose the number  $g_i$  of all the possible states of any one of these subsystems which have energy near  $\epsilon_i$  is much larger than the actual the number of subsystems  $n_i$  carrying those nearby energies, and that  $n_i$  itself is large. Then near thermodynamic equilibrium the number of subsystems with energy near  $\epsilon_i$  is given by  $n_i \propto g_i \exp(-\epsilon_i/(kT))$ , where  $T$  is the temperature of the system.

Interactions from the outside can change the total energy,  $E = \sum_i n_i \epsilon_i$ , of a system either by changing the ‘occupation numbers’  $\{n_i\}$ , and/or by changing the energies  $\{\epsilon_i\}$  of the quantum states. The first kind of change is heat transfer and the second is work transfer. By increasing the multiplicity of the system, putting heat into a system is a ‘disordering process’. Work involves changing the particle energies by changing the volume of the system, without moving particles between quantum states.<sup>63</sup> These ideas incorporate the first and second law of thermodynamics.

In terms of information, the second law of thermodynamics implies that *if two systems interact, each with fixed volumes, then that system of the two which has the smaller variation in its information content as its total energy changes will tend to spontaneously transfer information into the second system.*<sup>64</sup>

These are important concepts for quantum computers, as there is an intimate connection between entropy, information, decoherence, wave function collapse, and heat from memory loss.

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<sup>62</sup>Ludwig Boltzmann in “Über die beziehung dem zweiten Hauptsatze der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wärmegleichgewicht”, *Sitzungsberichte der Akademie der Wissenschaften zu Wien* **76**, pp. 373-435 (1877).

<sup>63</sup>If particles remain in their quantum states, no heat is transferred, and the process is called ‘adiabatic’.

<sup>64</sup>If a system near thermal equilibrium is held at fixed volume and a small amount of energy  $dE$  is put in, causing an increase in its information content by  $dI$ , then the ratio  $dE/dI$  turns out to be proportional to the temperature of that system. The spontaneous flow of information, i.e. non-forced flow, results from statistical likelihood.

## 6 Quantum computation

Feynman<sup>65</sup> considered the possibility that we might take advantage of quantum systems to perform computations quicker than so-called classical computers. Modern classical computers use bistable systems to store information, and logical gates to perform Boolean operations on sets of ones and zeros.<sup>66</sup> For some problems involving numbers with  $n$  digits and that may require solution times that rise exponentially with  $n$  when performed on computers using only Boolean logic, the computation on a quantum computer may take times that rise no faster than a power of  $n$ . Below are some of the special consequences of quantum theory for quantum computers and communications:

The simplest system for the storage of information gives only two possible values by a measurement. These values can be taken as 0 or 1, in which case the states are called  $|0\rangle$  and  $|1\rangle$ . Classically, such a system stores one bit of information. A quantum system can be constructed that has only these two values for the outcome of a measurement, but whose quantum state is a linear combination of the two possible outcome states  $|0\rangle$  and  $|1\rangle$ :

$$|q\rangle = \alpha |0\rangle + \beta |1\rangle.$$

This state is called a ‘qubit’, where  $\alpha$  and  $\beta$  are complex numbers satisfying  $|\alpha|^2 + |\beta|^2 = 1$ . An alternative parameterization takes  $\alpha = \cos(\theta/2)$  and  $\beta = \exp(i\phi) \sin(\theta/2)$ . Evidently, the possible qubit states can be pictured as points on a unit sphere (called the ‘Bloch sphere’) with  $|0\rangle$  at the north pole and  $|1\rangle$  at the south. Two-valued qubit states are easily realized in nature: The electron spin has only two possible projection values  $\pm 1/2\hbar$ , and the photon has only two possible helicity values  $\pm 1\hbar$ .

As it is always possible to expand an arbitrary quantum state into a basis set for that state’s Hilbert space,  $N$ -particle states in a quantum computer can be made by constructing these quantum states from a linear combination of the states for each of the  $N$  particles. Taking these particles to have only two

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<sup>65</sup>Richard Feynman, “Simulating Physics with Computers”, *International Journal of Theoretical Physics* **21**, pp. 467-488 (1982).

<sup>66</sup>These discrete-level computers are often referred to as ‘digital’, in contrast to ‘analog’ computers that use internal signals that are assumed to vary smoothly with time. Mechanical computers, which work by the movement and interaction of shaped objects, and molecular computers, that work through molecular interactions and transformations, are a mixed breed. The phrase digital computer, referring to counts base ten, can now mean any device which manipulates information by discrete changes. These days, the changes are made in systems which can flip between *off* and *on* in a specified clock time, i.e. a binary coding. By using such switching to encode information, digital computers can be more tolerant of a small amount of noise than analog devices. Shannon and Hartley showed that the maximum number of bits per second that can be transmitted from one storage location to another is given by  $B \log_2(1 + S/N)$ , where  $B$  is the bandwidth (in cycles per second),  $S$  is the average signal power, and  $N$  is the average noise power. See R.V. L. Hartley, “Transmission of Information”, *Bell System Technical Journal* (July 1928); C.E. Shannon, “Communication in the presence of noise”, *Proc. Institute of Radio Engineers* **37** (1), pp.10-21 (January 1949).

internal quantum states, the state of the computer is expressible by

$$\begin{aligned}
|\psi_N\rangle &= \sum_{\{i_k=0,1\}} \beta_{i_1 i_2 i_3 \dots i_N} |i_1\rangle_1 |i_2\rangle_2 |i_3\rangle_3 \cdots |i_N\rangle_N \\
&= \sum_{i=1}^{2^N} \beta_i |i_1 i_2 i_3 \cdots i_N\rangle \text{ with } \sum_i |\beta_i|^2 = 1.
\end{aligned} \tag{11}$$

In the second line of the equation, the product base state is represented in a shortened form, in which the order of the 0's and 1's corresponds to the labeling of each of the separate qubits, and  $i = i_1 i_2 i_3 \cdots i_N$  is a binary number constructed from the  $i$ 's. If the quantum state  $|\psi_N\rangle$  cannot be factorized, it harbors entanglement. Quantum computation takes advantage of entanglement within those states.<sup>67</sup> It follows that a useful initial state of a quantum computer has at least a subset of particles prepared in one of the **maximally entangled states**, i.e. states with equal probability for all possible configurations of its component particles, making it also that state which has maximum information content.<sup>68</sup> The maximally entangled states made from qubits as in Eq. (11) will have all  $|\beta_i|^2 = 1/N$ , leaving  $(2^N - 1)$  free relative phases between the basis states.<sup>69</sup>

To sustain coherence, quantum computers must operate on the input information stored in quantum states by unitary transformations. In the following, the substage of a quantum computation holding the intermediate state of a calculation will be called  $|\psi(k)\rangle$ , where  $k$  labels a particular intermediate state, with  $k = 0$  labeling the initial state. For a given quantum computer, a solution to a solvable problem is a unitary transformation  $U_S$  that carries the input quantum state  $|\psi(0)\rangle$  encoding the required initial data into an output quantum state that carries the information about the solution, at least in probabilistic terms. To be a non-classical computer, at least some the intermediate states must be entangled. It is possible that  $U_S$  can be decomposed into a finite product of simpler or more universal unitary operations:  $U_S = \prod_i U_{g_i}$ , where the set  $\{U_{g_i}\}$  are called ‘quantum gates’, a generalization of classical logic gates. Each term in the product acts on the state  $|\psi(k)\rangle$  left by the previous operation labeled by  $k$  and produces  $|\psi(k+1)\rangle$ .

Since a general unitary transformation will contain continuous parameters,  $U_S$  might only be approximated by a finite sequence of quantum gates. In the classical case, all Boolean operations on a set of bits can be performed by a combination of NAND gates. This makes NAND gates universal for classical computing. The same is true of NOR gates. In the quantum case, there are ‘universal’ sets of simple gates that can be used to build arbitrarily close representations of a general unitary transformation, such as  $U_S$ . (Arbitrarily close

<sup>67</sup>See, for example, Richard Jozsa and Noah Linden, “On the role of entanglement in quantum computational speed-up”, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **459** (2036), pp. 2011-2032 (2002).

<sup>68</sup>One learns most when the outcomes are least predictable!

<sup>69</sup>These maximally entanglement states are also called ‘generalized Bell states’.

here means that if  $V_S$  is the approximation, then  $|\langle\psi|(U_S - V_S)|\psi\rangle|^2$  is a number that can be made arbitrarily small for all  $|\psi\rangle$  by increasing the number of universal gates used in  $V_S$ .)

Quantum gates acting on a single qubit can all be represented by a general unitary transformation  $U_{\vec{\theta}}$  which is an arbitrary rotation in Hilbert space:

$$U_{\vec{\theta}} = \exp(i\theta \hat{n} \cdot \vec{\sigma}/2) = \cos(\theta/2) I + i\hat{n} \cdot \vec{\sigma} \sin(\theta/2),$$

where  $\theta$  is an angle of rotation around an axis fixed by the direction  $\hat{n}$ , and the  $\{\sigma_i ; i = 1, 2, 3\}$  are the Pauli matrices,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which act on the base states  $|0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and  $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . For an initial state consisting of the many qubits (perhaps realized by many particles capable of being in two distinct quantum states), such as  $|i_1 i_2 i_3 \dots i_N\rangle$ , a  $2N$  dimensional unitary transformation would be implemented to carry out one step of a computation.

If noise or other spurious interactions occur in the system, quantum coherence may be degraded or lost, and there will be both a ‘coherence time’ and a ‘coherence length’ over which the system retains a semblance of its coherence. A fault-tolerant quantum computer uses states that have long coherence times, quantum entangled states with long life times, and/or error correcting schemes. Systems for transferring qubits over long distances require long coherence lengths.<sup>70</sup>

A new measurement acting on a quantum state generally causes some decoherence, so that a number of components of the wave function may have their phase become stochastically indeterminate. The observation of the state of a particle in a multi-particle entangled state removes the entanglement of that particle. As we have seen, measurement of an observable is the equivalent of projecting out a subspace of the initial state:  $\psi_o = P_o \psi$ . Such a projection into a proper subspace is irreversible and non-unitary. The resulting state of the system no longer holds information about the complement  $(1 - P_o)\psi$  state.

In quantum theory, all processes within an isolated system preserve the condition that the probability of finding any of the possible states of the system add to unity. Formally, quantum states evolve by a unitary transformation. In the ‘Copenhagen’ view, the act of measurement causes the wave function for the system to ‘collapse’. A collapse of a quantum state from a superposition of substates to one such substate violates unitarity, and therefore is outside

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<sup>70</sup>Transferring qubits across space was first described by C. H. Bennett *et al.* in “Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels”, *Phys.Rev. Lett.* **70**, pp.1895-1899 (1993). Note that transferring a qubit from one system to another does not violate the no-cloning theorem, because the initial qubit is destroyed in the process, and that the transfer is cannot be superluminal, as two classical bits must be sent from the first system to the second before reconstruction of the qubit can take place.



the formalism of quantum theory. This produces a paradox: The measuring instrument is also a physical system, so that the larger system that contains the observed system and the measuring devices, left unobserved, should evolve by a unitary transformation, and no wave function ‘collapse’ should occur. There is no easy way out of this paradox.

The measuring devices in the larger combined system must introduce interactions that do not project out quantum substates in the combined system, but rather redistribute the amplitudes for various quantum states, making the observed state highly probable, and the other possible states in the observed system left with very small amplitudes. Being unitary for the combined system of the observed and the measuring device, such a measurement process is, in principle, ‘reversible’. The entanglement of an observed system with a measurement instrument and subsequent restoration of the original quantum state has been demonstrated for simple systems and measuring devices with highly restricted interactions.<sup>71</sup> But for a multitude of interactions, restoration after interactions is typically unfeasible with our current resources.<sup>72</sup> It is also possible that nature does not just scatter information so much that we cannot easily put systems such as broken eggs back together again, but rather actually does lose information over time. This possibility is outside the realm of quantum theory.

The same ideas apply to quantum computers. In quantum theory, even the measurement of a final state after a computer calculation is a reversible process for the computer, the measurement device and the surrounding interacting systems. In principle, no information is lost. But if the information transferred by erasing a quantum memory state produces heat in the environment, some information is practically lost.

If one of the particles in an entangled state is sent to a second observer as a form of communication, then attempts to intercept that particle will degrade or destroy the entanglement, and therefore will be detectable. This opens the possibility of ‘absolute’ security in transmission lines, particularly since macroscopically long coherence lengths have been realized with laser beams.

If a set of identical particles are restricted to a two-dimensional surface, or the space is not simply connected, the quantum state representing two particles may gain a phase factor of  $\exp(2i\pi p)$  when the two particles are exchanged, where  $p$  need not be integer or half-integer.<sup>73</sup> If the phase factor  $p$  is not  $n/2$  (where  $n$  is integer), the particles are called ‘anyons’. For three particles, if the order in which the particles are exchanged produces a different wave function phase, the group of such exchanges is non-abelian. This consideration may be important in the construction of quantum computers through the storing of information in the topological braiding of non-abelian anyons as they progress

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<sup>71</sup>See, for example, Nadav Katz, *et al.*, “Reversal of the Weak Measurement of a Quantum State in a Superconducting Phase Qubit”, *Phys.Rev.Lett.* **101**, 200401 (2008).

<sup>72</sup> This difficulty is related to the ergodic hypothesis in classical mechanics, and the development of entropy concept in statistical thermodynamics.

<sup>73</sup>By contrast, in a connected region of three dimensional space, there is a space transformation that will ‘untangle’ the pair, and make  $p$  an integer multiple of  $1/2$ .

in space-time.

Topological structures have been shown to be important in quantum theory. For example, the continuity condition for the wave function describing particles adds significance to global space-time topology. In some models, particle charges come from topological structures. A variety of promising systems for quantum computers take advantage of the difficulty of breaking topological structures in order to preserve quantum coherence, making the system more ‘fault immune’ against the effect of noise and other external interactions.

## 7 Limits to computing

### 7.1 Practical limits to computing and information storage

Classical computers have practical limitations in density. Gates and memory elements smaller than nanoscale will suffer quantum fluctuations, with growing uncertainties in bit structures and Boolean transformations as the size of the elements are reduced. Even our DNA code can be mutated by quantum tunneling. If the system has a certain level of noise, classical correction schemes can eliminate errors, at a cost of size. The techniques to control heat buildup also require volume in the ancillary heat sinks or channels for radiative cooling. Taking systems at the nanoscale and finding technology that minimizes heat production toward the Szilárd value of  $kT \ln 2$  per bit lost gives an upper limit to computer density made from materials. Memory and gates based on information in light beams have corresponding limits due to pulse duration and wave length uncertainties.

Quantum computers require coherence within the involved quantum states of the computer during computation. Working against us are physical limitations. For example, the quantum states being used to store information typically have finite lifetimes through spontaneous decay, resulting in the collapse of the employed coherent states. Uncontrollable interactions both within and from the outside a quantum computer will tend to collapse coherent states. After sufficient time, coupling to the environment will cause decoherence and disentanglement within a quantum system.

Coherence can be maintained for some period of time by using quantum states which have some intrinsic stability and suffer little debilitating interactions with adjacent systems or with the environment. Explorations to find strategies which minimize the limitations are ongoing. Evidently, each quantum gate must act within the shortest coherence time. Some mixing and degradation in quantum states can be tolerated by using repeated calculations and/or implementing error corrections which can reconstruct, with some assurance, a degraded quantum state. Overall, even though we can anticipate severe practical difficulties to building a quantum computer which can outperform its classical cousin, we see no fundamental limitation, unless our ambitions reach across the cosmos.

## 7.2 Cosmological limits

Strong gravitational fields exist near black holes, which are predicted by Einstein's General Theory of Relativity to occur when the density of an object of mass  $m$  exceeds about  $3c^6/(2^5\pi G^3 m^2)$ . Such black holes got their name because no form of radiation can escape from the hole if it starts out within a region around the hole bounded by a surface called the 'horizon'. For a non-spinning hole without charge, this surface has the 'Schwarzschild radius'<sup>74</sup>  $R_S \equiv 2Gm/c^2$ . Astronomers have found stellar-mass black holes in binary systems by analyzing the orbits of companion stars. Nearby large galaxies are known to contain one or more super-massive black holes at their center, and we suspect all large galaxies do.

Using quantum theory, Hawking showed<sup>75</sup> that the fluctuations in particle fields near but outside the horizon of a black hole can produce particle pairs with some of the positive energy particles having sufficient kinetic energy to reach large distances away, while the negative energy particles fall into the black hole. Thus, quantum theory requires that black holes evaporate, with a mass loss rate inversely proportional to the square of the hole mass  $m$  [ $dm/dt = -\hbar c^4/(3 \cdot 5 \cdot 2^{10} \pi G^2 m^2)$ ]. The flux of photons emitted is close to that of a hot body at a temperature inversely proportional to  $m$  [ $T = \hbar c^3/(8\pi k G m)$ ].

However, to be consistent with quantum theory, a system initially containing an object and a black hole, with the object destined to disappear into the black hole, with no other interaction but gravity, cannot lose information: The quantum state of the hole and the object evolves unitarily. One resolution of this paradox is to have the object's information transferred to a region close to the horizon of the black hole.<sup>76</sup> In this way, Hawking radiation can carry the stored information back out (so the radiation is not perfectly thermal). Even before Hawking proposed that black holes evaporate, Jacob Berkenstein<sup>77</sup> conjectured that the entropy of a black hole, which is also the information storage capacity, is proportional to the area of the hole's horizon,  $4\pi R_S^2$ , and inversely proportional to the square of Planck's length. Hawking then calculated the proportionality constant to be  $k/4$ , where  $k$  is Boltzmann's constant.

General Relativity limits the density of a computer, and concurrently the density of information storage. As a computer becomes larger in a given volume, its density eventually forces the computer to collapse into a black hole. This leads to the idea that the limiting density of information storage may be effectively two dimensional, with each bit stored in a Planck-size area. Some (as yet untested) theories even have the information of the whole Universe reflected by a kind of holographic image in one less dimension.

<sup>74</sup>Karl Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie", *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* **1**, pp. 189-196 (1916).

<sup>75</sup>S.W. Hawking, "Black hole explosions?", *Nature* **248** (5443), pp.30-31 (1976)

<sup>76</sup>It is even possible that the volume surrounded by a black hole horizon is completely empty, even of any space-time structure, with any infalling matter ending up just outside the horizon.

<sup>77</sup>J. D. Bekenstein, "Black holes and entropy", *Phys.Rev.* **D 7**, pp. 2333-2346 (1973).

A cosmological limitation on computation also comes from the fact that we appear to live in a finite Universe. A computer can be no larger than the Universe itself. Any smaller computer cannot hold the data of the Universe at one time, which is needed to unambiguously project the Universe's future. In addition, being that the computer is within the Universe, it cannot predict both itself and the Universe. Our current theories do not incorporate these kinds of limitations, although there are propositions that connect the very small to the very large.

## 8 Conclusions

Quantum computers take advantage of quantum operations in physical systems in order to solve well-posed problems. Quantum theory describes these operations based on how nature processes information. Space and time are important primitives in quantum theory, and active participants in both information transfer and information storage. While we formulate how nature handles information, we should recognize that our physical theories are always tentative. Each covers a limited realm and has a limited accuracy. Also, since each theory has a variety of equivalent formulations, with their own language, our main focus should be on the predictions of a theory. Even though very successful, quantum theory makes some rather non-intuitive and thought-provoking predictions. Correspondingly, there are a variety of precautions to which we should be attentive when applying and interpreting the theory. Reflecting on the underlying ideas central to quantum theory should help us in the exploration of possibilities for future quantum computers.